Homework
EVOLVING DOCUMENT
IISc Mech. Eng., Applied Dynamics I

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1 Problems

1. Set up (define system, draw FBD, write ODEs) a particle problem. Just one particle. 2D or 3D, your choice. Use a force, or forces that you like (gravity, spring, air friction). Any example of interest. Find a numerical solution. Graph it. Animate it. Try to make an interesting observation.

2. **3D. Get good at vectors.** Assume that the positions relative to an origin of four random points, which are randomly located in space are given as $\vec{\boldsymbol{r}}_{\mathrm{A}}, \vec{\boldsymbol{r}}_{\mathrm{B}}, \vec{\boldsymbol{r}}_{\mathrm{C}}$ and $\vec{\boldsymbol{r}}_{\mathrm{D}}$. Assume force $\vec{\boldsymbol{F}}$ is given. For each problem below write a single vector formula (one for each problem) that answers the question.

   a) The points A and B define an infinite line. So do the points C and D. Find the distance between these two lines. 'The' distance means 'the minimum distance', that is the length of the shortest line segment connecting the two lines. Either write a formula (or sequence of formulas), or write computer code that gives the answer, or both.

   b) Same problem as above, but also find the end points of the shortest line segment.

   c) Find the volume of the tetrahedron ABCD (you should reason-out and not quote any formulas for the volume of a tetrahedron, that is, see if you can derive the formula: 'volume = one third base times height').

   d) Assume points A, B and C are fixed to a structure. All three are connected, by massless rods, to a ball and socket at each end, to point D. At point D the force $\vec{\boldsymbol{F}}$ is applied. Find the tension in bar AD. Find a formula for the answer, or write computer code to find the answer, or both. The goal is to find a formula for the tension in terms of the positions and the force vector.

3. **More ODE & animation practice.** Take a simple set of ODEs. Use a set you like,*e.g.,* harmonic oscillator, non-linear pendulum, the Lorentz system (look it up on the internet). Solve this set numerically 3 ways (see below), and understand the accuracy. The goal is that, by the time you hand in the homework, you can write and debug the assignment on your own without looking up anything (outside of trivial syntax things). And you always have a good sense of the accuracy of your solution.

   a. Method 1: as simply as possible, without ODE45, and without calling functions or anything like that. A single function or script file with no function calls (ok, plotting calls are ok). Just write a simple loop that implements Euler's method with your ODE.

b. With your own Euler solver function. Your main program should call your Euler solver. Your Euler solver should call a RHS (Right Hand Side) function.
c. With `ODE45`.
d. Using (b), solve the equations many times with progressively smaller step size, down to the smallest size you have patience for, and up to the largest size that isn't crazy. As sensibly as possible, compare the results and use that comparison to estimate the accuracy of each solution. You should be able to find a method to estimate the accuracy of a numerical solution even without knowing the exact solution.
e. Using `ODE45`, solve the equations with various accuracies (use 'reltol' and 'abstol', note MATLAB satisfies one or the other, whichever is easiest. So, if you want an accurate solution you need to make both 'reltol' and 'abstol' small). Does Matlab do a good job of estimating its own accuracy? Use suitable plots to make your point.

4. **Cross product: geometry vs components.** The geometric definition of cross product is this $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$ is a vector $\vec{\boldsymbol{c}}$ with magnitude $|\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|\sin\theta_{ab}$ that is orthmorogonal to $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ in the direction given by the right hand rule. Use this definition to find an alternative geometric definition involving projection (namely: project $\vec{\boldsymbol{b}}$ onto the plane that is orthogonal to $\vec{\boldsymbol{a}}$; then stretch it by $|\vec{\boldsymbol{a}}|$; then rotate it $\pi/2$ around the $\vec{\boldsymbol{a}}$ axis). Use that definition to show the distributive rule $\vec{\boldsymbol{a}} \times (\vec{\boldsymbol{b}} + \vec{\boldsymbol{c}}) = \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} + \vec{\boldsymbol{a}} \times \vec{\boldsymbol{c}}$. b) Then use the distributive rule to find the component formula for cross product, namely that

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = (a_2b_3 - a_3b_2)\hat{\boldsymbol{e}}_1 + (a_3b_1 - a_1b_3)\hat{\boldsymbol{e}}_2 + (a_1b_2 - a_2b_1)\hat{\boldsymbol{e}}_3.$$

Note that, this distributive law implies that, for given va, $\vec{\boldsymbol{a}}\times$ is a linear operator. That is $va \times \vec{\boldsymbol{v}}$ is a linear function fo $\vec{\boldsymbol{v}}$. Later in the course we will use this to replace the cross product with a tensor product. **Hint:** You can read about this in, say, the Ruina/Pratap book (box 1.7).

5. **Read up.**

a. Read all of the course Teams posts so far, trying out all possible links of interest.
b. Make sure you thoroughly understand these sections of the Ruina/Pratap pdf book, available from Ruina's www page:
*Chapter 1 (read),
*Chapter 2 (skim, only study things you don't know well already, make sure that within a few weeks you know all of this *well.*),
*Chapter 3,
*Section 17.1,
*Appendix A.
c. Write the following, if true, if not write that which is true:

> *"I have carefully done some of the reading assigned. That which I don't understand or agree with, I have posted on the course Piazza site. I haven't yet carefully read X* [make appropriate substitutions for X]*".*

6. **Spring and mass (2D).** One end of of a negligible-mass spring (k, L_0) is pinned to the origin, the other to a mass (m). There is gravity (g). Initial Conditions (ICs):

The initial position is $\vec{\boldsymbol{r}}_0 = x_0\hat{\boldsymbol{\imath}} + y_0\hat{\boldsymbol{\jmath}}$, and the initial velocity is $\vec{\boldsymbol{v}}_0 = v_{x0}\hat{\boldsymbol{\imath}} + v_{y0}\hat{\boldsymbol{\jmath}}$. Motion starts at $t = 0$ and ends at t_{end}.

(a) Find the Equations of Motion (EoM);

(b) Assume all parameters and IC's above are given.

   i. Plot the trajectory of the mass.

   ii. Animate the trajectory of the mass.

(c) How many ways can you think of checking the numerical solution, find as many as you can, and do the check. The list is started here:

   i. $k = 0$, all else arbitrary: The motion is parabolic flight (including falling straight down as a special case) [Why? The system is then just ballistics from freshman physics];

   ii. $x_0 = 0, v_{x0} = 0$, all else arbitrary: The motion stays on the y axis [Why? There is no force in the x direction if the mass is on the y axis. Because the initial velocity has no x component, the mass never leaves the y axis;

   iii. $g = 0, \vec{\boldsymbol{v}}_0 = \vec{\boldsymbol{0}}$, all else is arbitrary: The motion stays on a radial line. And, if the motion does not cross the origin, the motion is that of a harmonic oscillator (sinusoidal oscillations, check by plotting, say x vt t. [Why? Write the EoM and EoMs in polar coordinates
   $\Rightarrow \quad m\ddot{r} = -k(r - L_0) \quad \Rightarrow$ the harmonic oscillator equation, $m\ddot{r}^* = -kr^*$, where $r^* \equiv r - L_0$.

   iv. $L_0 = 0$, all else is arbitrary: ___?___. [Why? ___?___.] Hint, this one special case is problem 10, below.

   v. *etc.*

   vi. *etc.*

   vii. ...

7. **Simple animation of a shape.** Draw a picture of some object (a face, a house, whatever), and make it move around on the screen in a smooth and interesting way. No distortions. Just motions and rotations.

8. **Simplest dynamics with Polar coordinates.** This is the simplest dynamics problem, but posed in polar coordinates. Assume a particle is on a plane with no force on it. So, you know it moves at constant speed in a constant direction.

   a. Write the differential equations
   $$\vec{\boldsymbol{a}} = \vec{\boldsymbol{0}}$$
   in polar coordinates.

   b. Solve them numerically for various initial conditions.

   c. Plot the solution and check that the motion is a straight line at constant speed.

   d. Using your numerical result, pick a way to measure how straight the path is, and see how straight a line your polar coordinate solution gives. You should define a quantitative measure of straightness, and then measure it with your solution.

   e. Is the path more straight when you refine the numerical tolerances.

9. **Canon ball.** A cannon ball m is launched at angle θ and speed v_0. It is acted on by gravity g and a viscous drag with magnitude $|cv|$.

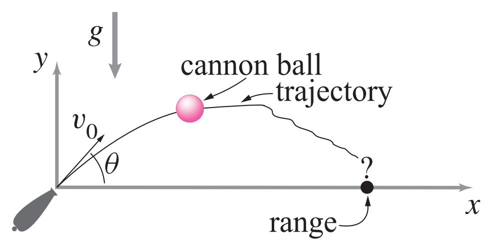
(a) Find position vs time analytically.

(b) Find a numerical solution using $\theta = \pi/4$, $v_0 = 1\,\mathrm{m/s}$, $g = 1\,\mathrm{m/s^2}$, $m = 1\,\mathrm{kg}$, $c = 1\,\mathrm{kg/\,s}$.

(c) Compare the numeric and analytic solutions. At $t = 2$ how big is the error? How does the error depend on specified tolerances or step sizes?

(d) Use larger and larger values of v_0 and for each trajectory choose a time interval so the canon at least gets back to the ground. Plot the trajectories (using equal scale for the x and y axis. Plot all curves on one plot. As $v \to \infty$ what is the eventual shape? [Hint: the answer is simple and interesting.]

(e) For any given v_0 there is a best launch angle θ^* for maximizing the range. As $v_0 \to \infty$ to what angle does θ^* tend? Justify your answer as best you can with careful numerics, analytical work, or both.

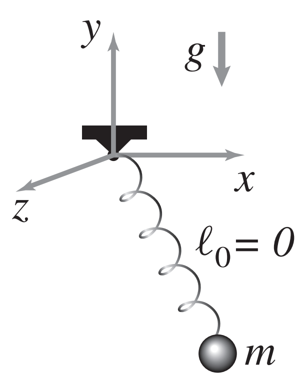


10. **Mass hanging from spring.** 3D. Consider a point mass hanging from a zero-rest-length linear spring ($\ell_0 = 0$) in a constant gravitational field.

(a) Set up equations. Set up for numerical solution. Plot 2D projection of 3D trajectories.

(b) By playing around with initial conditions, find the most wild motion you can find (wild means most wiggles, or most complicated). Make one or more revealing plots. [Hint: Make sure the features you observe are properties of the system and not due to numerical errors. That is, check that the features do not change when the numerics is refined.]

(c) Using analytical methods justify your answer to part (b).



11. **Central force.** These two problems are both about central forces. In both cases the only force is central (directed on the particle towards the origin) and only depends on radius: $\vec{\boldsymbol{F}} = -F(r)\hat{\boldsymbol{e}}_r$. The problems are independent, one does not follow from the other.

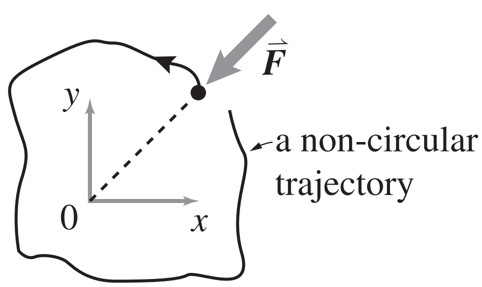
(a) Find a central force law $F(r)$ so that, comparing circular orbits of varying radii, the speed v is independent of radius.

(b) By numerical experiments, and trial and error, try to find a periodic motion that is *neither* circular nor a straight line, for some central force *besides* $F = -kr$ or $F = -GmM/r^2$. a) not with a linear zero-rest length spring; b) not with inverse-square gravity; c) not circular motion; and d) not straight-line motion.

  i. In your failed searches, before you find a periodic motion, do the motions always have regular patterns or are they sometimes chaotic looking (include some pretty pictures)?

  ii. Puzzle: If you use a power law, what is the minimum number loops in one complete periodic orbit (a loop is, say, a relative maximum in the radius)? How does this depend on the exponent in the power law? You probably cannot make progress with this analytically, but you can figure it out with numerical experiments.

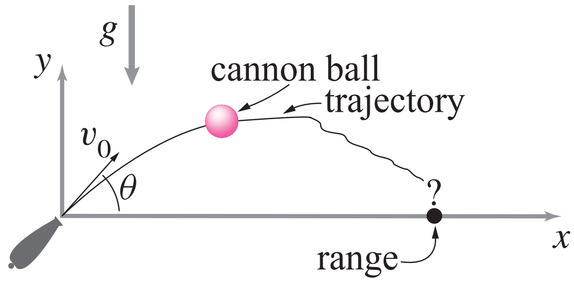
[Matlab hint: To do this properly you probably need to guess at a radial force law (most anything will work) and do numerical root finding (*e.g.,* `FSOLVE`) to find initial conditions and the period of the orbit. Once you have your system you can define a function whose input is the initial conditions and the time of integration and whose output is the difference between the initial state and the final state. You can make this system 'square' by assuming that the particle is on the x axis in the initial state. You want to find that input which makes the output the zero vector. Pick a central force and search over initial conditions and durations. Do *not use* `FSOLVE` to search over force laws; do your orbit finding using a given force law.



12. **Canon ball 2.** A cannon ball m is launched at angle θ and speed v_0. It is acted on by gravity g and a quadratic drag with magnitude $|cv^2|$.

(a) Find a numerical solution using $\theta = \pi/4$, $v_0 = 1\,\mathrm{m/s}$, $g = 1\,\mathrm{m/s^2}$, $m = 1\,\mathrm{kg}$.

(b) Numerically calculate (by integrating $\dot{W} = P$ along with the state variables) the work done by the drag force. Compare this with the change of the total energy. Make a plot showing that the difference between the two goes to zero as the integration gets more and more accurate.



13. **Statics vs Dynamics.** 2D. Equilibrium and dynamics of a point mass m at $\vec{\boldsymbol{r}}_C$. Gravity g points down in the minus $\hat{\boldsymbol{\jmath}}$ direction. Points A and B are anchored at $\vec{\boldsymbol{r}}_A$ and $\vec{\boldsymbol{r}}_B$. Springs and parallel dashpots connect the mass to A and B. The spring constants are k_A and k_B. Their rest lengths are L_A and L_B. Parallel to the springs are dashpots c_A and c_B. The mass also has a drag force proportional to speed through the air c_d.

    The dynamic state of the system is described with $z = [x, y, \dot{x}, \dot{y}]'$ and the static state by $z = [x, y, 0, 0]'$.

    (a) **Basic Statics.** Given parameters and an initial guess, find an equilibrium position of this system. If you have an example you like, post it and people can compare solutions.

    (b) **Basic Dynamics.** Given parameters and an initial condition, find the motion.

    (c) **All equilibria.** Given parameters, attempt to find all equilibria. How many are there? By varying parameters, what is the most and least number of equilibrium points there can be?

    (d) **Stability of equilibrium.** Given an equilibrium point, find it's stability 2 ways: a) With a dynamic simulation near the equilibrium; and b) looking at the eigenvalues of the matrix describing the linearized equations of motion. Compare.

    Make appropriate plots, animations and comparisons.

14. **What means "rate of change of angular momentum"?** Consider a moving particle P. Consider also a moving point C (moving relative to a Newtonian frame $\mathcal{F}$ that has an origin 0). For which of these definitions of $\vec{\boldsymbol{H}}_{/\mathrm{C}}$ Is the following equation of motion true (that is, consistent with $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$)?

    $$\vec{\boldsymbol{M}}_{\mathrm{C}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$$

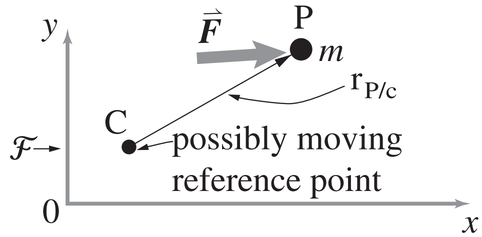
    In each case say whether the definition works i) in general, or ii) for some special cases (that you name) concerning the motions of P and C.

    (a) $\vec{\boldsymbol{H}}_{/C} = \vec{\boldsymbol{r}}_{\mathrm{P/C'}} \times \vec{\boldsymbol{v}}_{\mathrm{P/0}} m$,
    where C' is a point fixed in $\mathcal{F}$ that instantaneously coincides with C.

    (b) $\vec{\boldsymbol{H}}_{/C} = \vec{\boldsymbol{r}}_{\mathrm{P/C}} \times \vec{\boldsymbol{v}}_{\mathrm{P/0}} m$.

    (c) $\vec{\boldsymbol{H}}_{/C} = \vec{\boldsymbol{r}}_{\mathrm{P/C}} \times \vec{\boldsymbol{v}}_{\mathrm{P/C}} m$.

    That is, for each possible definition of $\vec{\boldsymbol{H}}_{/\mathrm{C}}$ you need to calculate $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$ by differentiation and see if and when you get $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} = \vec{\boldsymbol{r}}_{\mathrm{P/C}} \times m\vec{\boldsymbol{a}}_{\mathrm{P}/\mathcal{F}}$.



15. **Potential Energy** For each of the force fields below, find the associated potential energy function. All of them are 3D. In some cases the energy is associated with the positions of two particles. Any free constant can be set to a convenient value (like zero). The answers are easy, it sometimes takes careful multi-variable calculus to find or check them.

    (a) Constant force $\vec{\boldsymbol{F}} = -mg\hat{\boldsymbol{\jmath}}$. (Ans: $E_p = mgy$)

    (b) Zero-rest-length spring, one end at origin $\vec{\boldsymbol{F}} = -kr\hat{\boldsymbol{e}}_r = -k\vec{\boldsymbol{r}}$. (Ans: $E_p = kr^2/2$)

    (c) Spring, one end at origin $\vec{\boldsymbol{F}} = -k(\ell - \ell_0)\hat{\boldsymbol{e}}_r$. (Ans: $E_p = k(\ell - \ell_0)^2/2$)

    (d) Inverse square central force $\vec{\boldsymbol{F}} = -C\vec{\boldsymbol{r}}/r^3$. (Ans: $E_p = -C/r$)

    (e) Central force, depending on r, $\vec{\boldsymbol{F}} = -f(r)\hat{\boldsymbol{e}}_r$. (Ans: $\int_{r_0}^{r} f(r')\,dr'$, with r_0 chosen so the integral is not divergent)

    (f) Spring between two points. (Ans: $E_p = k(\ell_{12} - \ell_0)^2/2$)

    (g) Inverse square attraction between two points. (Ans: $E_p = -C/\ell_{12}$)

    (h) Constant force $\vec{\boldsymbol{F}} = -C\hat{\boldsymbol{\lambda}}$. (Ans: $E_p = C\vec{\boldsymbol{r}} \cdot \hat{\boldsymbol{\lambda}}$)

    (i) Force in x direction only depending on x, $\vec{\boldsymbol{F}} = -f(x)\hat{\boldsymbol{\imath}}$. (Ans: $\int_{x_0}^{x} f(x')\,dx'$, with x_0 chosen so the integral is not divergent)

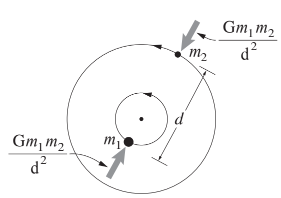
16. **Mechanics of two or more particles**

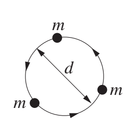
    (a) For two unequal particles with mass m_1 and m_2 what is the period of circular motion if the distance between the particles is d and the only force is the force between them is from classical inverse-square Newtonian gravity, $F = Gm_1m_2/r^2$?

    (b) Pick numbers for G, m_1, m_2 and r and, using appropriate initial conditions, test your analytical result with a numerical simulation. Make any plots needed to make the agreement of your numerical and analytical results convincing.

    (c) For three equal particles, $m_1 = m_2 = m_3 = 1$ and $G = 10$ what is the angular speed for circular motion on a circle with diameter of $d = 3$?

    (d) Check your result for three particles with a numerical simulation, as you did for 2 particles, above.





17. **Montgomery's eight.** (From Ruina/Pratap). Three equal masses, say $m = 1$, are attracted by an inverse-square gravity law with $G = 1$. That is, each mass is attracted to the other by $F = Gm_1m_2/r^2$ where r is the distance between them. Use these unusual and special initial positions:

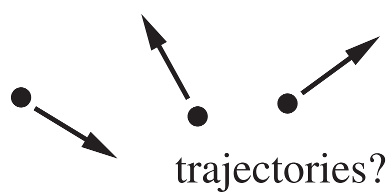
$$\begin{aligned}(x1, y1) &= (-0.97000436, 0.24308753)\\ (x2, y2) &= (-x1, -y1)\\ (x3, y3) &= (0, 0)\end{aligned}$$

and initial velocities

$$\begin{aligned}(vx3, vy3) &= (0.93240737, 0.86473146)\\ (vx1, vy1) &= -(vx3, vy3)/2\\ (vx2, vy2) &= -(vx3, vy3)/2.\end{aligned}$$

For each of the problems below show accurate computer plots and explain any curiosities.

(a) Use computer integration to find and plot the motions of the particles. Plot each with a different color. Run the program for 2.1 time units.
(b) Same as above, but run for 10 time units.
(c) Same as above, but change the initial conditions slightly.
(d) Same as above, but change the initial conditions more and run for a much longer time.



[Aside: This was discovered by both Richard Montgomery (Santa Cruz Math department) and also by ex-Cornell Physics PhD student Chris Moore (now at Santa Fe Institute), independently. And, I know both of them, independently.]

18. **Konig's Theorem** The total kinetic energy of a system of particles is

$$E_{\rm K} = \frac{1}{2}\sum m_i v_i^2.$$

(a) Derive an expression of this form

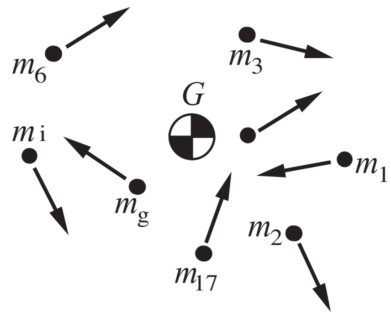
$$E_{\rm K} = \frac{1}{2} m_{\rm tot} v_G^2 + \text{...you fill in the rest....}$$

(b) Is it always true that

$$\left(\sum \vec{\boldsymbol{F}}^{ext}\right)\cdot \vec{\boldsymbol{v}}_{\rm G} = \frac{d}{dt}\left(\frac{1}{2} m_{\rm tot} v_G^2\right)?$$

Defend your answer with unassailable clear reasoning (that is, a proof or a counter-example).

(c) Is it always true that the power of internal forces is equal to the rate of change of the quantity you filled in part (a) above (just the second half of the full expression)? Provide a proof or a counter-example. (A good solution is expected from those in 5730).



19. **What means "rate of change of angular momentum" for a SYSTEM of particles?** You should master problem 14 before looking at this problem. Consider a system of moving particles with moving center of mass at G. Consider also a moving point C (moving relative to a Newtonian frame $\mathcal{F}$ that has an origin 0). For which of these definitions of $\vec{\boldsymbol{H}}_{/\mathrm{C}}$ Is the following equation of motion true (that is, consistent with $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$)?

$$\vec{\boldsymbol{M}}_{\mathrm{C}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$$

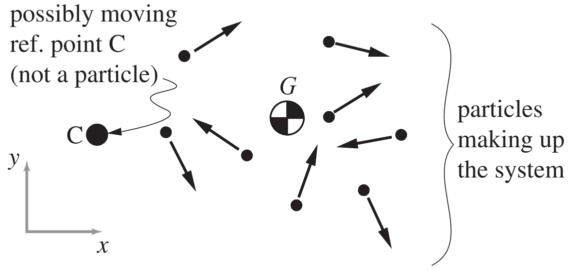
In each case say whether the definition works i) in general, or ii) for some special cases (that you name) concerning the motions of P and C.

(a) $\vec{\boldsymbol{H}}_{/C} = \sum \vec{\boldsymbol{r}}_{\mathrm{i/C'}} \times \vec{\boldsymbol{v}}_{\mathrm{i/C'}} m_i$,
where C' is a point fixed in $\mathcal{F}$ that instantaneously coincides with C.
(Hint: this definition is good one, always!)

(b) $\vec{\boldsymbol{H}}_{/C} = \sum \vec{\boldsymbol{r}}_{\mathrm{i/C}} \times \vec{\boldsymbol{v}}_{\mathrm{i/0}} m_i$.
(This strange definition is used in the classic, but in this case odd, Dynamics book by Housner and Hudson.)

(c) $\vec{\boldsymbol{H}}_{/C} = \sum \vec{\boldsymbol{r}}_{i/C} \times \vec{\boldsymbol{v}}_{i/C} m_i$.
(Hint: this is the most important candidate definition, but it's only good for special kinds of C, namely: C = COM, C is fixed and ...?)

That is, for each possible definition of $\vec{\boldsymbol{H}}_{/\mathrm{C}}$ you need to calculate $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$ by differentiation and see if and when you get $\sum \vec{\boldsymbol{r}}_{\mathrm{i/C}} \times \vec{\boldsymbol{a}}_{\mathrm{i/0}} m_i$. If you are short for time just consider cases (a) and (c) and note their agreement if C is stationary or if C=G. Students in 5730 are expected to do a competent clear job of all three parts.



20. **Two masses** This problem has 3 independent educational goals:

    (a) Motivate the use of kinematic constraints.

    (b) Introduce the simplest of a class of vibrations problems which you should master. At this point, it is this aspect: mastery of derivation of the equations of motion. You should check that you can reproduce the lecture example with *no* sign errors without looking up anything.

    (c) Development of critical exploration using analytical and numerical methods.

    Two masses m_1 and m_2 are constrained to move frictionlessly on the x axis. Initially they are stationary at positions $x_1(0) = 0$ and $x_2(0) = \ell_0$. They are connected with a linear spring with constant k and rest length ℓ_0. A rightwards force is applied to the second mass. It is a step, or '**Heaviside**' function

$$F(t) = F_0 H(t) = \begin{cases} 0 & \text{if } t < 0 \\ F_0 & \text{if } t \geq 0 \end{cases}$$

    (a) Write code to calculate, plot and (optionally) animate the motions for arbitrary values of the given constants.

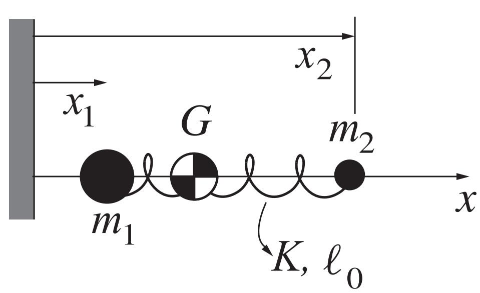
    (b) Within numerical precision, should your numerical solution always have the property that $F = (m_1 + m_2)a_G$ where $x_G = (x_1 m_1 + x_2 m_2)/(m_1 + m_2)$? (As always in this course, "yes or no" questions are not multiple choice, but need justification that another student, one who got the opposite answer as you, would find convincing.)

    (c) Use your numerics to demonstrate that if k is large the motion (displacement relative to its starting position) of *each* mass is, for time scales large compared to the oscillations, close to the center of mass motion.

    (d) 5730 only: Using analytic arguments, perhaps inspired by and buttressed with numerical examples, make the following statement as precise as possible:

    For high values of k the system nearly behaves like a single mass.

    Of course, in detail, the system has 2 degrees of freedom (DOF). So you are looking for a way to measure the extent to which the system acts like it has 1 DOF, and in which conditions (for which extreme values of parameters and times) the system is close to 1 DOF by that measure. There is not a simple single unique answer to this question.



21. **Two masses constrained** This is an elaboration of the problem above, replacing the spring with a rigid rod. As per lecture, set up the DAEs and solve them using Matlab

using numbers of your choice. Use a forcing F that varies sinusoidally in time. Note the increasing error (as time progresses) in the satisfaction of the constraint. Compare this solution with the the method from the problem above (where you use some very large value of k). Which one is is faster to compute on the computer? Which is more accurate in predicting COM motion? (Again, as always, justify your answer so someone who disagrees would be convinced.) Try putting in initial conditions in which the masses have unequal initial velocity; explain as best you can the result you get, and why.

22. **Simple pendulum.** Derive the simple pendulum equation $\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$ as many ways as you can without looking anything up in books. For example, in all cases using polar coordinates,

    (a) linear momentum and manipulate the equations to eliminate constraint force
    (b) linear momentum, dot with $\hat{\boldsymbol{e}}_\theta$
    (c) linear momentum, cross with $\vec{\boldsymbol{r}}$
    (d) angular momentum
    (e) conservation of energy
    (f) power balance
    (g) Lagrange equations (if you know them already, not if you don't).

23. **Pendulum numerics.** Set up the pendulum in cartesian coordinates. Express the constant length constraint as a set of linear equations restricting the acceleration. Solve these (3 2nd order) DAE equations with numerical integration and initial conditions and parameters of your choosing. No polar coordinates allowed. Quantitatively compare your solution with a solution of the simple pendulum equations (For the comparison you need to either compute x from θ or *vice versa*. Integrate for a long enough time so you can detect drift away from satisfying the kinematic constraint.

24. **Pendulum with an awkward parameterization** By any means you like, for a simple pendulum find the equations of motion using y (horizontal position) as your parameterization of the configuration. That is, find a 2nd order differential equation determining $\ddot{y}$ in terms of y, $\dot{y}$ and physical parameters (g, m, ℓ). Using numerics, quantitatively compare the solution of this ODE with a solution of the simple pendulum equations (Note, you can assume the pendulum is hanging down, hence $x > 0$. This problem is different from the DAE problem above in that you should obtain a single 2nd order ODE, not a set of 3 equations.

25. **2D Dumbell.** Two equal masses $m = 1$ are constrained by a rod to be a distance $\ell = 1$ apart. There are no non-negligible forces, except from the constraint. At $t = 0$ they have equal and opposite velocities ($v = 1$) perpendicular to the rod. Use a set of

5 DAEs ($\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ & the constraint equation $(x_2 - x_1)^2 + (y_2 - y_1)^2 = \ell^2$ with numerical integration to find the subsequent motion. Use plots and/or animation to help debug your code. Using what you know about systems of particles (*e.g.,* momentum, angular momentum, constraint equation, energy) quantify as many different numerical errors as you can.

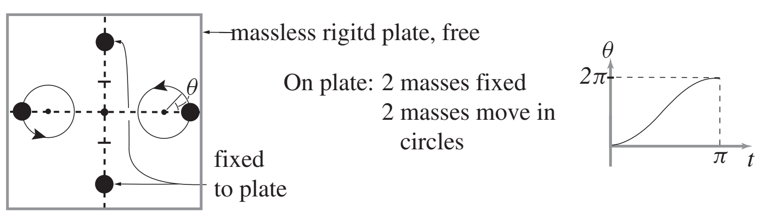
26. **Rotation with zero angular momentum.** This is a concrete example showing that a system can rotate while having zero angular momentum at all times. If you don't like this example, or want to do a different one for any reason, any substitution is fine. But it needs to be backed by a clear (numerical is fine) calculation.

    Consider a massless rigid plate in space. On this plate are marked x and y axes. There are 4 equal masses attached to the plate. Two of them are welded (glued, fixed) to the plate on the y axis at $y = \pm 2$. Two of them are forced, by massless machinery that rides on the plate, to move in a circle with radius $r = 1$ centered at $x = \pm 2$, and starting at $x = \pm 3$, respectively. They move such that the line connecting them goes through the origin at all times. The right mass moves according to this equation, using polar coordinates centered at $y = 0, x = 2$:

    $$r = 1, \qquad \theta = (1 - \cos(t))\,\pi \quad \text{for} \quad 0 \le t \le \pi.$$

    That is, the mass goes around the circle once in a nice smooth motion. The other mass moves accordingly (same motion, reflected through the origin).

    Use conservation of angular momentum for the system to find the net rotation ϕ of the masses on the y axis at the end of one rotation of the $x-$ axis masses. Numerical integration is fine. [Hint: The equation $\dot{\vec{\boldsymbol{H}}}_{/0} = \vec{\boldsymbol{0}}$ tells you $\dot{\phi}(t)$.]



27. **Canonball.** (Continuation of problem 10) A cannon fires a projectile that experiences gravity and quadratic drag $F_D = c_D v^2$). Cannonball mass $m = 1\,\text{kg}$, acceleration due to gravity $g = 10\,\text{m/s}^2$, and drag coefficient is $c_D = 0.5\,\text{kg/ m}$.

    i. Find a trajectory (using whichever numerical tool you choose, but don't just guess!) so that the projectile hits the ground at $x = 2\,\text{m}$ given using a launch speed of $v_0 = 10\,\text{m/s}$. Find the launch angle and a plot of the trajectory.

    ii. Imagine you want an efficient cannon that can hit a point at least $2\,\text{m}$ away with the minimum possible launch speed. Find the angle and launch velocity that achieve this.

28. Reconsider the Montgomery 8 problem again (3 point masses mutually attracted by gravity) using $m_i = G = 1$. Find a periodic solution by optimizing both the initial conditions and simulation time so that the error between initial and final state (positions and velocities) goes to 0. Afterwards simulate for 5+ periods to demonstrate that your solution is periodic. The following initial conditions are a reasonable guess:

Mass 1: X1=-0.755; Y1=0.355; Vx1=0.9955; Vy1=0.07855;

Mass 2: X2=1.155; Y2=-0.0755; Vx2=0.1055; Vy2=0.4755;

Mass 3: X3=-0.4055; Y3=-0.3055; Vx3=-1.1055; Vy3=-0.5355;

A good initial guess for the period is 8 (8 time units) . Hint: You'll be most successful if you bound your search near to these values. If you don't use variable bounds (e.g. in FMINCON), you'll probably find the trivial solution where everything equals 0. If you want to go on with this problem, you can try it again using Matlab's BVP4C command (which is said to do this particular problem more elegantly).

Etra: See if you can find any other periodic solutions of the 3-body problem.

29. **Inverted pendulum with shaking base.** A uniform stick with length ℓ is connected to a hinge at its lower end. That hinge is shaking up and down with a specified acceleration $a(t)$ (could be a specified position, differentiated twice).
    (a) Find the equations of motion.
    (b) Simulate the motion to show that the pendulum can be stable when upright. [Hint: Use an oscillating base with a small displacement amplitude ($\delta \ll \ell$) and a big enough frequency ($a \gg g$).]
    (c) Do again with DAEs, and show that your results quantitatively agree.

30. **Bead on parabolic wire** For a frictionless point-mass bead sliding on a rigid wire on the curve $y = cx^2$ with gravity in the $-y$ direction, find the equation of motion.
    (a) Derive the equations of motion using the x axis as the minimal (generalized) coordinate. Use Newton's laws.

31. **Double pendulum. 2D.** A double pendulum is made with two bars. Hinges are at the origin 0 and the elbow E. $g = 1$. Neglect all friction and assume there are no joint motors.
    (a) Set up and numerically solve (there is no analytic solution) to the governing equations that you find using AMB. You may refer to lecture notes, but you should be able to do it on your own by the time you hand in the work. Assume that at $t = 0$ the upper arm is horizontal, sticking to the right, and the fore-arm is vertical up (like looking from the front at a driver using hand signals to signal a right turn). Integrate until $t = 30$. Draw the (crazy) trajectory of the end of the forearm. For definiteness (and so you can check solutions against each other) use parameters corresponding to equal-length uniform bars with length 1 and use $g = 1$.
    (b) Solve again by DAEs equations. By comparing numerical solutions, show that the governing equations are the same as those obtained using AMB.
    (c) And again with Lagrange equations, and compare again.
    (d) Animate your solution. Put your solution in a folder. Name the folder: YOURNAME-Double. In the folder have a function called ROOT (and other well-commented Matlab files). Running ROOT should give an animation within 30s. Zip the

folder. Call the zipped file YOURNAME-Double. Have that ready to send to your TA.

(e) Extra things you can do:

  i. Explore various options for setting up the ODEs. For example: cut and paste from symbolic toolbox, Jacobian, EquationsToMatrix. When using DAEs: extract the minimal coordinate angular accelerations vs consider the motion of the independent rigid objects (6 DoF system).
  ii. Use various checks: Compare methods by subtracting one result from the other; if applicable, check energy conservation or angular momentum conservation by plotting the value, subtracting the initial value.
  iii. Think of as many special cases as you can where you know something about the solution without doing a numerical solution.
  iv. make your animation and plotting and nice as you can.
  v. *etc.*

32. **Braking stability** 2D, looking down. Consider the steering stability of a car going straight ahead with either the front brakes locked or the rear brakes locked. The steering is locked and straight ahead. For simplicity assume that the center of mass is at ground height between the front and back wheels. Assume that the locked wheels act the same as a single dragging point on the centerline of the car, midway between the wheels.

   (a) Develop the equations of motion.

   (b) Set them up for computer solution. Note: ODE45 is generally unhappy when the car comes to a stop (because of a divide by v in your code). One way to avoid this is by using the 'events' option to stop the calculation when the speed gets down to some very small number.

   (c) For some reasonable parameters and initial conditions find the motion and make informative plots that answer the question about steering stability. Note, in this problem where there is no steady state solution you have to make up a reasonable definition of steering stability.

   (d) See what analytical results you can get about the steering stability (as dependent on the car geometry, mass distribution, the coefficient of friction and the car speed). As much as you have time and interest, illustrate your results with graphs and animations of numerical integrations.

   (e) **Things to think about.**

      i. Check that your governing equations reduce to the lecture (ideal, no friction) equations when the friction is zero.
      ii. Check special cases of the numerical solutions with solutions you know other ways. A challenge is to think of as many of these as you can (even if you don't check all of them). That is, for some special parameter values and/or initial conditions you know features of the solution (examples:
         A. No friction means energy is conserved;
         B. With friction and no initial rotation slowing is with constant acceleration,
         C. *etc.*

33. **Bead on parabolic wire, Lagrange Equations** For a frictionless point-mass bead sliding on a rigid wire on the curve $y = cx^2$ with gravity in the $-y$ direction, find the equation of motion.

(a) Derive the equations of motion using the x axis as the generalized coordinate. Check that this agrees with the Newton-Euler approach.

(b) A curve in the xy plane is described by the following equation:

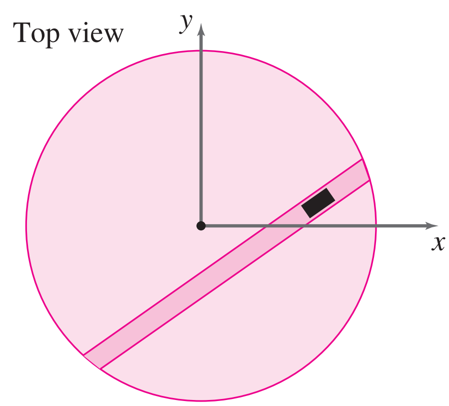
$$y = cs^2,$$

where s is arc-length along the curve $y = cs^2$, starting at $x = y = 0$ (x is only defined implicitly on this curve, using the equation $ds = \sqrt{(dx)^2 + (dy)^2}$. A bead slides on that wire with gravity. Use Lagrange equations to find the EoM. [Hint: this is a really easy problem].
Extra credit:

i) draw the curve (probably requires numerical integration or ODE solution),

ii) find a more normal description for the curve (e.g., a parametric equation), [Hint: It is a famous curve, with a special name and all.]

34. **Mass in slot on turntable.** A rigid turntable (m_t, I_t) is free to rotate about a hinge at it's center. It has in it a straight frictionless slot that passes a distance d from it's center. A mass m_s slides in the slot. For minimal coordinates use rotation of the disk θ from the position in which the slot is horizontal and below the disk center, and the distance s the mass is from the point where the slot is closest to the center of the disk.



(a) Find the acceleration of the mass in terms of $d, \theta\dot{\theta}, \ddot{\theta}, s, \dot{s}$ and $\ddot{s}$. Extra credit: Do this one or two more different ways and check that all give the same answer when reduced to x and y coordinates.

i. Write the position of the mass in terms of d, θ and s using base vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$. Differentiate twice.

ii. Write the position using $\hat{\boldsymbol{\imath}}'$, which aligns with the slot, and $\hat{\boldsymbol{\jmath}}'$. Differentiate twice using that $\dot{\hat{\boldsymbol{\imath}}}' = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\imath}}' = \omega\hat{\boldsymbol{\jmath}}'$ and $\dot{\hat{\boldsymbol{\jmath}}}' = -\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\jmath}}' = -\omega\hat{\boldsymbol{\imath}}'$

iii. Use the five-term acceleration formula (using $\vec{\boldsymbol{v}}_{\text{rel}} = \dot{s}\hat{\boldsymbol{\imath}}'$ and $\vec{\boldsymbol{a}}_{\text{rel}} = \ddot{s}\hat{\boldsymbol{\imath}}'$).

(b) Find the equations of motion (That is, find $\ddot{\theta}$ and $\ddot{s}$ in terms of fixed parameters and position and velocity variables.). Do this using AMB for the whole system about the center and {LMB for the mass}$\cdot\hat{\boldsymbol{\imath}}'$.

(c) Find the equations of motion using Lagrange equations, and check for agreement.

(d) Assume ICs that $s(0) = 0, \dot{s}(0) = v_0, \theta(0) = 0$ and $\dot{\theta}(0) = \omega_0 > 0$. As $t \to \infty$ does $\theta \to \infty$? (As for all questions, please explain in a way that would convince a non-believer.)

35. **Cart and pendulum** A cart m_1 slides frictionlessly on a level surface. A massless stick with length ℓ is hinged to it with a mass m_2 at the end. Take $\theta = 0$ to be the configuration when the pendulum is straight down. Use gravity g.

    (a) Find the full non-linear governing equations using Newton-Euler (N-E, Linear and angular momentum balance).

    (b) and using Lagrange. Show that you get the same equations as for N-E.

36. **Double pendulum kinematics.** Consider a double pendulum where both links have length ℓ. Consider two sets of minimal coordinates:

    (a) Absolute angles θ_1, θ_2. These are the angles from the vertical of the two links.

    (b) Relative angles $\theta_1, \phi = \theta_2 - \theta_1$.

    Calculate the acceleration of the end of the second link in terms of the two sets, and their derivatives. (This is *kinematics* only. There is no place here for force, moment, free-body diagrams, mass, linear momentum, angular momentum, or energy.)

    - For the first set do this, as we have done before, by considering a non-rotating frame that moves with the end of the first link. This involves basic polar coordinate formulas.
    - For the second set, consider a reference frame that is glued to the first link. Then use the '5-term acceleration formula' to calculate the acceleration of the end of the second link.

    Show that the two methods give the same answer.

37. **Rolling cylinder** A uniform cylinder with mass m and radius r rolls without slip outside and on top of a fixed cylinder with radius R. Gravity g pulls it down.

    (a) Are the full non-linear differential equations the same as those of an inverted pendulum? If so, or not, explain why this is expected.

    (b) Assume the rolling cylinder is released from rest almost exactly at the top: $\theta = 0^+$. At what angle will it lose contact. Find an analytic formula for this [Hint: use energy conservation.] Extra: if you like, compare the result with a numerical solution of the ODEs using 'events'

38. **Rolling eccentric cylinder** 5730 students only. A cylinder with radius R has center of mass G offset from the cylinder center C by a distance $d < R$. It has total mass M, radius R and moment of inertia I^G about it's center of mass. It rolls without slip down a ramp with slope γ (angle relative to horizontal), propelled by gravity g.

    (a) Find the equations of motion.

(b) Find the needed coefficient of friction to enforce the rolling constraint.

(c) After release from rest, assuming it starts with the line between CoM and contact point orthogonal to the surface, how far does it roll before it skips into the air.

It's ok to use numerical solutions based on any non-trivial parameter choices. No need for parameter sweeps.

Interesting extension if you have *lots* of time: With appropriate initial conditions, this can roll on a level ramp then skip, then do about a full revolution in the air, then land with no no relative velocity at impact (thus conserving energy) and continue rolling and skipping. This is explained, somewhat, in a paper on Ruina's www page: "A collisional model ..." (figure 4 there in). The calculation details are like those in this paper "Persistent Passive Hopping ..." on Ruina's www page. I would love someone to build a model of this. Talk with me if you are interested.

39. **Two skates.** A rigid object (m, I^G) moves on the plane with no gravity or other forcing. It has on it two skates in arbitrary position on the object and at arbitrary orientations, with one exception.

- **Special case to avoid:** The two skates have a common normal line. This happens when in the two skates are parallel to each other, and are both orthogonal to the line connecting them. In this one, special, excluded situation, the two skates would be redundantly applying the same kinematic constraint. Further, in this configuration, because the two reaction forces have the same line of action, there is no dynamics or statics equation for the object which has other than the sum of the two forces; you can only find their sum, the difference between the reaction forces is indeterminate.
- **All other cases are fine:** All other skate configurations (angles) are ok, only the one above is problematic.

(a) Do this problem from beginning to end with just one skate. Do not attempt two skates until you have mastered the case of a single skate. If that is as far as you get, hand in your work for a single skate.

(b) Find the equations of motion using the DAE approach. In this approach you have a 2D rigid object with constraint forces from the skate that have associated kinematic constraint equations.

(c) Find the motion by writing unconstrained Lagrange Equations (3 DoF rigid object on a plane). Then calculate generalized forces associated with the constraints. Then supplement the Lagrange equations with constraint equations (two times over, once for each skate: write the skate velocity constraint and differentiate it). These are DAEs coming from Lagrange equations. Then solve these equations numerically using example parameters of your choosing.

(d) The solution above can also be found by simple elementary means. If you don't see this, make an animation of the solution above. Compare the simply reasoned result, with the numerical solution.

Hints:

- Again, do not try this problem until you can do it all, from writing equations through to animation, for a mass supported by just one skate;

- Your DAEs only enforce the constraint at the acceleration level. Your initial velocities need to be consistent with the kinematic constraints.
- Your DAEs should be the same with Newton-Euler as with Lagrange.]

40. **Practice with dyadics.** Read as much as you like, and do the exercises in **Paul Mitiguy**'s Stanford course. (Hand in a sentence about the effort you put into this.)

41. **Index notation & Einstein Summation convention.** For dealing with vector and tensor components and base vectors we often have one of these situations

$$d = \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} \qquad d = \sum_{i=1}^{3} a_i b_i \qquad \text{dot product of two vectors}$$

$$[\vec{\boldsymbol{c}}] = A[\vec{\boldsymbol{b}}] \qquad [\vec{\boldsymbol{c}}] = \sum_{i=1}^{3} A_{ij} b_j \qquad \text{matrix times column vector}$$

$$C = AB \qquad C_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj} \qquad \text{matrix times matrix}$$

where a_i, b_i and c_i are the components of vectors $\vec{\boldsymbol{a}}$, $\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{c}}$; $[\vec{\boldsymbol{a}}]$, $[\vec{\boldsymbol{b}}]$ and $[\vec{\boldsymbol{c}}]$ are column vectors of their components; and A_{ij}, B_{ij} and C_{ij} are the components of 3×3 matrices A, B and C.

SUMMATION CONVENTION

Einstein, apparently (because the rule is named after him), noticed that in these sums, the summed index always happens to occur twice in the multiplicative expression. That's i in the first two examples above and k in the third. So he made up a rule to save himself the trouble of writing '$\sum_{i=1}^{3}$'. Here's the rule

> **If a subscript appears twice in a multiplicative expression then adding up (summing), from one to three, is implied.**

So, repeating the three examples above, using the summation convention we have that,

$$d = a_i b_i \quad \text{means} \quad d = \sum_{i=1}^{3} a_i b_i \qquad \text{dot product of two vectors}$$

$$c_i = A_{ij} b_j \quad \text{means} \quad [\vec{\boldsymbol{c}}] = \sum_{j=1}^{3} A_{ij} b_j \qquad \text{matrix times column vector}$$

$$C_{ij} = A_{ik} B_{kj} \quad \text{means} \quad C_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj} \qquad \text{matrix times matrix.}$$

According to these rules

- You must write "summation convention" before the first equation, for example

  > **From this point on in this course, unless stated otherwise, we are using the summation convention.**

- The sum is always from 1 to 3 unless you are doing a strictly 2D problem, in which case the sum is from 1 to 2;

- If you have a repeated subscript in a multiplicative expression and you don't want to sum, you have to write 'no sum' next to the equation, for example

  $$c_i = a_i b_i \qquad \text{(no sum)}$$

  means that $c_1 = a_1 b_1$, $c_2 = a_2 b_2$, and $c_3 = a_3 b_3$ which is quite different than, using the summation convention.

  $$c = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

FREE INDICES and DUMMY INDICES
There are free indices and dummy indices. In the formula

$$c_i = A_{ij} b_j$$

i is a free index and j is a dummy. Free indices survive the equation and dummies are killed off. The equation above has no j in it. Well, it looks like it's there, but it's private and not revealed in the final evaluation. On the other hand, i is free and could take on any value, from, usually, 1 to 3. And it survives. Thus, the equation above is really three equations,

$$c_1 = A_{1j} b_j \quad \text{and} \quad c_2 = A_{2j} b_j \quad \text{and} \quad c_3 = A_{3j} b_3$$

Rules. You can change any index to a different letter according to these rules:

- If it is a free index you have to change it at every occurrence on both sides of the equation;
- If it is a dummy index, you have to change it at both occurences in any multiplicative expression.

For example, the equation

$$c_i = A_{ij} b_j + B_{ki} c_k \qquad \text{(Note: } i \text{ is free, and } j \text{ and } k \text{ are dummies)}$$

is equivalent to all of the following equations,

$$\begin{aligned} c_\ell &= A_{\ell j} b_j + B_{k\ell} c_k && \text{Note: change the free index } i \text{ to } \ell \\ c_i &= A_{ij} b_j + B_{ji} c_j && \text{Note: the two multiplicative expressions do not interact} \\ c_j &= A_{jk} b_k + B_{kj} c_k && \text{Note: a few switches.} \end{aligned}$$

You should check that you agree. If you assigned any 24 numbers to each of the components on the right hand side, you should get the same values for c_1, c_2 and c_3 in all 4 of the expressions above.

On the other hand, if you were to write

$$a_i = b_j$$

most likely you are talking nonsense. If you meant it, here's what the expression says. It says that $a_i = b_j$ for all values of the free indices i and j. This would mean that once you knew, say, a_1 all three of b_1, b_2, and b_3 would be equal to that. And then, in turn, so would a_2 and a_3. That is, the equation above would be saying that $a_1 = b_1 = b_2 = b_3 = a_2 = a_3$. Unless you intend a meaning like that, which you probably do not, the free index, or indices, on both sides of any equation need to match.

Rules of grammar:

Every multiplicative expression on both sides of the equation have to have the exact same free variables. For example, if there's a free i and ℓ on the left, then there should be a free i and ℓ on the right.

Dummies can be recycled on one side of an equation and from one side to the other. For example, the following is sensible, with i being a dummy, three times over, and j a free variable.

$$A_{ii}b_j + f_i g_i h_j = B_{ii}e_j.$$

An index cannot occur more than 2 times in any multiplicative expression. For example,

$$c_i = b_i d_j e_j f_j \qquad \text{is nonsense}$$

NONSENSE
If you see nonsense, stop. Examples of nonsense:

- free indices are not the same on both sides of an equation;
- free indices are not the same in different multiplicative expressions on one side of an equation;
- an index occurs 3 or more times in a multiplicative expression
- unwittingly misusing the distributive law of multiplication:

  $a_i b_i = 14$ does *not* imply that $b_i = 14/a_i$. (Try it with $a_1 = b_1 = 1, a_2 = b_2 = 2, a_3 = b_3 = 3$ and note that, for example $b_1 \neq 14/a_1$);

  $R_{ij}R_{ij} \neq R_{ij}^2$. For starters, the left side has no free indices and the right has the 2 free indices i, j. So the expression is too wrong to even check for numerical equality. A scalar is not equal to an array.

SPECIAL SYMBOLS To aid calculations, here are two special arrays of numbers. First and simplest is the **Kronecker delta** which is just nine numbers, all of which are 1's and 0's.

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

That is, δ_{ij} are the components of the 3×3 identity matrix, with ones on the diagonal and zeros off of the diagonal. Here are some features of the Kronecker delta:

- $\delta_{ii} = 3$;
- $a_i \delta_{ij} = a_j$;
- $a_i b_j \delta ij = a_i b_i$;
- $\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}$;
- $\delta_{ik}\delta_{kj} = \delta_{ij}$.

Check all of these and make sure they are right.

Finally there is the Levi-Civita epsillon symbol, or 'alternating epsillon' symbol. This is 27 numbers all of which are 0, 1, or -1. Actually 3 of them are 1, three are -1 and all 21 of the rest of them are 0.

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if} \quad \text{ijk= 123, or 231, or 312} \quad \text{(even permutations of 123)} \\ -1 & \text{if} \quad \text{ijk= 213, or 132, or 321} \quad \text{(odd permutations of 123)} \\ 0 & \text{if} \quad i = j \text{ or } j = k \text{ or } k = i \quad \text{(not permutations of 123)} \end{cases}$$

The properties of ϵ_{ijk} are not so obvious as those for δ_{ij}. Here are three

- $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = ai\hat{\boldsymbol{e}}_i \times b_i\hat{\boldsymbol{e}}_j = \epsilon_{ijk}a_ib_j\hat{\boldsymbol{e}}_k$
- $[\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}] = \mathcal{S}([\vec{\boldsymbol{a}}])[\vec{\boldsymbol{b}}] = a^{\mathcal{S}}[\vec{\boldsymbol{b}}]$, where $[\vec{\boldsymbol{v}}]$ is a column vector of components of $\vec{\boldsymbol{v}}$ and $\mathcal{S}([\vec{\boldsymbol{a}}]) = a^{\mathcal{S}}$ is a skew-symmetric matrix with components $a^{\mathcal{S}}_{ij} = -\epsilon_{ijk}a_k$. That is, the cross product of two vectors can be calculated as the ordinary matrix product of an appropriate matrix with a vector.

Problems. Assume we have defined the 3×3 matrices A and B and column vectors a and b as:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \& \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Evaluate these expressions or declare them as non sensible and explain why.

(a) $c = a_ib_i$

(b) $c = \delta_{\ell\ell}$

(c) $c_i = A_{ij}b_j$

(d) $c_i = b_jA_{ij}$

(e) $c_i = A_{ji}b_j$

(f) $c = A_{ij}B_{ij}$

(g) $c = A_{ij}\delta_{ij}$

(h) $c_k = B_{jk}b_k$

(i) $c_i = A_{ij}B_{jk}$

(j) $C_{ik} = A_{ij}B_{jk}$

(k) $\delta_{ij}\delta_{jk}\delta_{ki}\delta_{\ell\ell}$

(l) $c_k = \epsilon_{ijk}a_ib_j$

42. **Rotations.** 3D rotations are hard to visualize. They have many representations: axis-angle, rotation vector, vector formula with cross products, dyadic representation of that vector formula, a tensor, a matrix, Euler angles (all n versions), Euler parameters and quaternians. The notations for each of these are not universal. And for each there are the rules for composing rotations. Add to this that the same word "Rotation", as for all linear transformations, can sometimes be thought of as a way

to change a vector and sometimes as a way to change a given vector's representation. Most people get confused about rotations some times.

We need to describe a vector, say $\vec{\boldsymbol{r}}$, it's components, say r_i and the name of a corresponding column vector of 3 numbers, say r. However, for all but the first of these we also need to know what basis we are using, say the original fixed basis $\mathcal{F}$ or a rotated basis $\mathcal{B}$. And the same issues show for the tensor, say $\underline{\underline{\mathbf{R}}}$ or $\vec{\vec{\mathbf{R}}}$, its components, say R_{ij} and the array of numbers used to represent it, say R. Then, we are interested in all the same things for the action of $\underline{\underline{\mathbf{R}}}$ on $\vec{\boldsymbol{r}}$, call it, say

$$\vec{\boldsymbol{r}}^* = \underline{\underline{\mathbf{R}}} \cdot \vec{\boldsymbol{r}}$$

[Aside: for this write up a vector $\vec{\boldsymbol{r}}$ gets rotated to become $\vec{\boldsymbol{r}}^*$. Another notation which we sometimes use is to start with a vector $\vec{\boldsymbol{r}}_0$ and rotate it to $\vec{\boldsymbol{r}}$.] Consider two frames, the fixed one $\mathcal{F}$ with an orthonormal basis $\hat{\boldsymbol{e}}_i$ and $\mathcal{B}$, which is $\mathcal{F}$ rotated by $\underline{\underline{\mathbf{R}}}$, with basis vectors $\hat{\boldsymbol{e}}'_i$. Here is a notation, including some free choices about how explicit to be about the choice of frame ($\mathcal{F}$ or $\mathcal{B}$).

NOTATION

$\hat{\boldsymbol{e}}_i =$ an orthonormal set of fixed $\mathcal{F}$ basis vectors.

$\hat{\boldsymbol{e}}'_i =$ the rotated $\mathcal{B}$ basis vectors (some people call these $\hat{\mathbf{b}}_i$).

$\underline{\underline{\mathbf{R}}} =$ the rotation, a linear transformation, that takes all vectors in $\mathcal{F}$ to $\mathcal{B}$. So

$$\underline{\underline{\mathbf{R}}} \cdot \hat{\boldsymbol{e}}_i = \hat{\boldsymbol{e}}'_i \tag{1}$$

$$\underline{\underline{\mathbf{R}}} \cdot \vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}^* \tag{2}$$

$$\tag{3}$$

Three popular representations of rotation are:

$$\begin{aligned} \underline{\underline{\mathbf{R}}} &= \hat{\boldsymbol{e}}'_i \hat{\boldsymbol{e}}_i && (4)\\ &= R_{ij} \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j && (5)\\ &= \hat{\boldsymbol{n}}\hat{\boldsymbol{n}} + \cos\theta(\underline{\underline{\mathbf{1}}} - \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}) + \sin(\theta)\underline{\underline{\mathbf{n}}} && (6) \end{aligned}$$

where $\hat{\boldsymbol{n}}$ is the axis of the net rotation and θ the amount of the rotation. The skew symmetric tensor $\underline{\underline{\mathbf{n}}}$ associated with $\hat{\boldsymbol{n}}$ is such that $\underline{\underline{\mathbf{n}}} \cdot \vec{\boldsymbol{r}} = \hat{\boldsymbol{n}} \times \vec{\boldsymbol{r}}$.

$R_{ij} = R_{\mathcal{F}ij} = R^{\mathcal{F}}_{ij} =$ various notations, progressively more formal, for the components of $\underline{\underline{\mathbf{R}}}$ in the fixed $\mathcal{F}$ basis. That is, if there is no $\mathcal{F}$ subscript or super script, it is assumed. But if you want to be clear that the same basis vectors are used for the left and right vectors in the dyads, then you use superscripts and subscripts. If you are never going to use mixed bases, then a single $\mathcal{F}$ will be good enough. Here are various ways of thinking about the rotation matrix.

$$\begin{aligned} R_{ij} &= \text{the } i\text{th component, in the } \mathcal{F} \text{ basis, of } \hat{\boldsymbol{e}}'_j && (7)\\ &= \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}'_j && (8)\\ &= \text{cosine of the angle between } \hat{\boldsymbol{e}}_1 \text{ and } \hat{\boldsymbol{e}}'_j && (9)\\ &= n_i n_j + \cos(\theta)(\delta_{ij} - n_i n_j) - \sin(\theta)\epsilon_{ijk} n_k && (10) \end{aligned}$$

$R_{\mathcal{B}ij} = R^{\mathcal{B}}_{ij} = R^{\mathcal{B}}_{\mathcal{B}ij} =$ various notations, progressively more formal, for the components of $\underline{\underline{\mathbf{R}}}$ in the rotated $\mathcal{B}$ basis. You need at least some $\mathcal{B}$ as a subscript or super script. But if you want to be super clear that the same basis vectors are used for the left and right vectors in the dyads, then you use superscripts and subscripts. If you are never going to use mixed bases, then a single $\mathcal{B}$ will be good enough.

$R_{\mathcal{F}ij}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j = \underline{\underline{\mathbf{R}}} =$ dyadic representation of $\underline{\underline{\mathbf{R}}}$ using $\mathcal{F}$ basis.

$R_{\mathcal{B}ij}\hat{\boldsymbol{e}}'_i\hat{\boldsymbol{e}}'_j = \underline{\underline{\mathbf{R}}} =$ dyadic representation of $\underline{\underline{\mathbf{R}}}$ using $\mathcal{B}$ basis.

$\vec{\boldsymbol{r}} = r_i\hat{\boldsymbol{e}}_i = r_{\mathcal{F}i}\hat{\boldsymbol{e}}_i = r_{\mathcal{B}i}\hat{\boldsymbol{e}}'_i =$ some vector of interest.

$\vec{\boldsymbol{r}}^* = \underline{\underline{\mathbf{R}}}\cdot\vec{\boldsymbol{r}} = r_{\mathcal{F}i}\hat{\boldsymbol{e}}'_i = r^*_{\mathcal{F}i}\hat{\boldsymbol{e}}_i = r^*_{\mathcal{B}i}\hat{\boldsymbol{e}}'_i =$ the rotation by $\underline{\underline{\mathbf{R}}}$ of $\vec{\boldsymbol{r}}$.

$R = R_{\mathcal{F}} = R^{\mathcal{F}} = R^{\mathcal{F}}_{\mathcal{F}} = [\underline{\underline{\mathbf{R}}}]_{\mathcal{F}} = [\underline{\underline{\mathbf{R}}}]^{\mathcal{F}} = [\underline{\underline{\mathbf{R}}}]^{\mathcal{F}}_{\mathcal{F}} =$ matrix representation of $\underline{\underline{\mathbf{R}}}$ using $\mathcal{F}$ basis.

$R_{\mathcal{B}} = R^{\mathcal{B}} = R^{\mathcal{B}}_{\mathcal{B}} = [\underline{\underline{\mathbf{R}}}]_{\mathcal{B}} = [\underline{\underline{\mathbf{R}}}]^{\mathcal{B}} = [\underline{\underline{\mathbf{R}}}]^{\mathcal{B}}_{\mathcal{B}} =$ matrix representation of $\underline{\underline{\mathbf{R}}}$ using $\mathcal{B}$ basis.

$r = r_{\mathcal{F}} = r^{\mathcal{F}} = [\vec{\boldsymbol{r}}]_{\mathcal{F}} = [\vec{\boldsymbol{r}}]^{\mathcal{F}} =$ matrix representation of $\vec{\boldsymbol{r}}$ using $\mathcal{F}$ basis.

$r^* = r^*_{\mathcal{F}} == [\vec{\boldsymbol{r}}^*]_{\mathcal{F}} = [\vec{\boldsymbol{r}}^*]^{\mathcal{F}} =$ matrix representation of $\vec{\boldsymbol{r}}^*$ using $\mathcal{F}$ basis.

$r = r_{\mathcal{B}} = r^{\mathcal{B}} = [\vec{\boldsymbol{r}}]_{\mathcal{B}} = [\vec{\boldsymbol{r}}]^{\mathcal{B}} =$ matrix representation of $\vec{\boldsymbol{r}}$ using $\mathcal{B}$ basis.

$r^* = r^*_{\mathcal{B}} == [\vec{\boldsymbol{r}}^*]_{\mathcal{B}} = [\vec{\boldsymbol{r}}^*]^{\mathcal{B}} =$ matrix representation of $\vec{\boldsymbol{r}}^*$ using $\mathcal{B}$ basis.

LOOK OUT FOR NONSENSE

If you start with a true expression, and then do legitimate operations to both sides of an equation, then you will write correct things. And with luck, a correct thing that you want. But if you start with nonsense, or do nonsense operations, you will get nonsense.

v_i, $[\vec{\boldsymbol{v}}]$ (sometimes written as just v), and $\vec{\boldsymbol{v}}$ are all in reference to a given vector $\vec{\boldsymbol{v}}$. But no two of them are equal to each other. One is a scalar, one a $s \times 1$ array, and one an abstract (or geometric) vector.

Similarly, R_{ij}, $[\underline{\underline{\mathbf{R}}}]$ (sometimes written as just R), and $\underline{\underline{\mathbf{R}}}$ are all in reference to a given tensor $\underline{\underline{\mathbf{R}}}$. But no two of them are equal to each other. One is a scalar, one a 3×3 array, and one an abstract (or geometric) tensor.

If one side of an equation has one of the 6 things above, and the other side a different one of the six things above, you have made an error.

QUESTIONS.

So that you get the general idea, the first few questions are answered for you. In all cases, try to derive the results from expressions that you solidly trust, like $\underline{\underline{\mathbf{R}}}\cdot\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}'_1$.

a) Given $\vec{\boldsymbol{r}}$ and the $\hat{\boldsymbol{e}}_i$ what is $[\vec{\boldsymbol{r}}]_{\mathcal{F}} =$ the column vector of the components of $\vec{\boldsymbol{r}}$ using the base vectors from $\mathcal{F}$? Answer: $[\vec{\boldsymbol{r}}]_{\mathcal{F}} = \begin{bmatrix} \vec{\boldsymbol{r}}\cdot\hat{\boldsymbol{e}}_1 \\ \vec{\boldsymbol{r}}\cdot\hat{\boldsymbol{e}}_2 \\ \vec{\boldsymbol{r}}\cdot\hat{\boldsymbol{e}}_3 \end{bmatrix}$.

b) Given $\underline{\underline{\mathbf{R}}}$ and the $\hat{\boldsymbol{e}}_i$ find R_{ij}? Answer: $R_{ij} = \hat{\boldsymbol{e}}_i\cdot\underline{\underline{\mathbf{R}}}\cdot\hat{\boldsymbol{e}}_j$.

c) Given the $\hat{\boldsymbol{e}}_i$ and the $\hat{\boldsymbol{e}}'_i$ find R_{ij}?

d) Given R_{ij} and r_i find r_i^*?

e) Given R_{ij} and r_i find $r_{\mathcal{B}i}^*$?

f) Given R_{ij} and r_i^* find r_i?

g) Given R_{ij} and r_i^* find $r_{\mathcal{B}i}$?

h) Given R_{ij} find $R_{\mathcal{B}ij}$?

i) Write $[\vec{\boldsymbol{r}}^*]_{\mathcal{F}}$ as many different ways as you can.

j) Given $\underline{\underline{\mathbf{A}}} = R_{ij}\hat{\boldsymbol{e}}_i'\hat{\boldsymbol{e}}_j'$, find A_{ij}. Note, $\underline{\underline{\mathbf{A}}} = A_{ij}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j$.

k) Pose and answer a few questions, of the general type above, that most test your remnent confusions.

43. **Final computation project.** Due at the end of the semester. This is an extension of the double pendulum homework. The minimal version is to simulate and animate both a triple pendulum and also a 4-bar linkage. For the triple-pendulum the equations of motion should be found three different ways. In both problems the numerical solutions should be checked as many ways as possible (Energy conservation, limiting cases where simple-pendulum motion is expected, etc). Optional extras are a) to simulate and animate more complicated mechanisms of your choice (e.g., 4,5, n link pendulum or closed kinematic loop or end link with a skate constraint) and b) to find periodic motions, c) to derive the 4-bar linkage equations by up to three methods (in total).

    Deliverables:

    (a) Send one zip file called YourName4730.zip or Yourame5730.zip.
        i. That should be a compressed version of a single folder.
        ii. In that folder should be a collection of Matlab files
        iii. In that folder should be a *README file explaining how to use the Matlab files. It should be VERY EASY to use the files for simple demonstrations
        iv. In that folder should be a file called REPORT.pdf. It could be made from WORD, LateX or scanned handwork, or any mixture of those. It should explain what you have done, how, and give sample output. This is the main demonstration of your effort. As appendices this should include your documented matlab files.
        v. A video presentation of your final project. Three minutes maximum.

    (b) Your assigned exam time will start with the examiner (Ruina) watching your 3 minute presentation movie.

44. Only for fun. Not assigned. Learn any non-algabraic "proof" of Euler's theorem that any "rotation" (any motion of a sphere with the center fixed) is a rotation about an axis. Once you have learned it, write it up neatly without looking at any source.

45. Consider a 90 degree rotation about the y axis followed by a 90° rotation about the x axis.

    a) Using geometric reasoning, find the net axis and angle of rotation as best you can.

b) Apply the axis-angle vector formula,

$$\vec{\boldsymbol{r}}' = \hat{\boldsymbol{n}}\ \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{r}} + \cos(\theta)(\vec{\boldsymbol{r}} - \hat{\boldsymbol{n}}\ \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{r}}) + \sin(\theta)\hat{\boldsymbol{n}} \times \vec{\boldsymbol{r}} \tag{11}$$

twice in sequence, to get the new positions of unit vectors that were originally on the x, y and z axis, respectively.

c) By taking the previous results as columns of a matrix, or by operating directly on a general vector, find the net rotation of any vector with original components (x, y, z).

46. Write some Matlab functions

a) `function rp = vecrot(n, theta, r)`

b) function R = rotmat(n, theta)

c) use `VECROT` and the matrix from `ROTMAT` to calculate the rotation for some special cases. Check against the answers you know from geometry.

i) 90° rotation about x axis,

ii) 90° rotation about z axis,

iii) 120° rotation about (1,1,1) axis.

d) Write a Matlab script that takes rotation matrix as input and calculates the axis and angle of rotation. Check that this works forwards and back with your code from above, both ways, with some odd examples.

47. Using `PLOT3`, or any other way of making 3D drawings (you can use objects, drawnow, patch, etc), draw a box with sides b,w, and h and a line through the box in direction n. Then, by repeatedly using PLOT3 rotate the box smoothly rotating about axis n. Use this to illustrate the 120° rotation about the axis $(1, 1, 1)$.

48. **Euler Angles.** Here is one set of Euler angles. The matrix representing rotation about the Fixed x, y and z axes are easy to construct. Call them $R^x(\theta_x), R^y(\theta_y)$ and $R^z(\theta_z)$. The net rotation of rotating about the x axis, then the fixed y axis then the fixed z axis is $R^{net} = R^z R^y R^x$. This is also the net rotation matrix if you rotate first about the z axis, then about the new y axis and then about the newest x axis. Use algebra, or computer algebra, to find R^{net}.

49. **Animation.** Draw any 3D object that pleases you in Matlab. A box would be good. Pick three functions of time for three Euler angles. Use anything interesting to you. Use `LINSPACE` to make three arrays of time. Make a look that uses your solution above to make a rotation matrix at each time, apply this to your drawing and thus animate your drawing moving in interesting ways. Debug your program by making the angles non-zero one at a time so you can see roll, pitch and yaw independently.

50. Given the basic postulate of angular momentum balance expressed as

$$\sum \vec{\boldsymbol{M}}_{/C} = \sum \vec{\boldsymbol{r}}_{i/C} \times \vec{\boldsymbol{a}}_{i/\mathcal{F}} m_i$$

show that

$$\sum \vec{\boldsymbol{M}}_{/G} = \frac{d}{dt}\vec{\boldsymbol{H}}_{/G} \qquad \text{where} \qquad \vec{\boldsymbol{H}}_{/G} = \sum \vec{\boldsymbol{r}}_{i/G} \times \vec{\boldsymbol{v}}_{i/G} m_i.$$

51. **Euler equations.** We found the equations for the general motion, at least the rotation part, of a rigid object:

$$\underline{\underline{\mathbf{I}}} = \underline{\underline{\mathbf{R}}} \cdot \underline{\underline{\mathbf{I}}}^* \cdot \underline{\underline{\mathbf{R}}}^t \tag{12}$$
$$\dot{\vec{\boldsymbol{\omega}}} = \underline{\underline{\mathbf{I}}}^{-1} \cdot \left(\vec{M}_{/G} - \vec{\boldsymbol{\omega}} \times \left(\underline{\underline{\mathbf{I}}} \cdot \vec{\boldsymbol{\omega}}\right)\right) \tag{13}$$
$$\dot{\underline{\underline{\mathbf{R}}}} = \underline{\underline{\mathbf{\Omega}}} \cdot \underline{\underline{\mathbf{R}}} \tag{14}$$

where $\underline{\underline{\mathbf{I}}}^*$ is the inertia in the reference state and $\underline{\underline{\mathbf{\Omega}}}$ is the skew-symmetric tensor version of $\vec{\boldsymbol{\omega}}$. The $\dot{\vec{\boldsymbol{\omega}}}$ is ambiguous, or interpretable two ways, because ${}^{\mathcal{F}}\dot{\vec{\boldsymbol{\omega}}} ={}^{\mathcal{B}} \dot{\vec{\boldsymbol{\omega}}}$. By expressing these tensors in different coordinate systems, and dotting with different base vectors, these equations can be turned into various ODEs in terms of various representations of $\vec{\boldsymbol{\omega}}$ and $\underline{\underline{\mathbf{I}}}$.

(a) Draw a box that is long in the x direction and narrow in the z direction so the moment of inertia in the reference configuration could plausibly be:

$$[I] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(b) Simulate and animate the torque-free motion of this box using the Euler equations and the initial condition $[\vec{\boldsymbol{\omega}}]^0_{\mathcal{F}} = [0.01,\ 1.0,\ \ 0.0]'$. Integrate for a long enough time that the motion is interesting. Consider any other problems you like.

52. **Euler equations.** These are the Euler equations. Not to be confused with the Euler eqautions above. Use the object-fixed coordinate system $\mathcal{B}$ based on $\hat{\boldsymbol{e}}'_i$ in which the moment of inertia matrix is

$$[I] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

a) Use the derivation of the Euler equations from lecture. These are three equations that give $\dot{\omega}_{1\mathcal{B}}, \dot{\omega}_{2\mathcal{B}}$ & $\dot{\omega}_{3\mathcal{B}}$ in terms of $\omega_{1\mathcal{B}}, \omega_{2\mathcal{B}}$ & $\omega_{3\mathcal{B}}$ and I_1, I_2 & I_3. You need not write the linearized equations.

b) Set up numerical solutions of the general non-linear free motion of such an object. That is, given initial values for the body-fixed angular velocity components, find the body-fixed angular velocities as a function of time.

c) As an initial condition, consider a small deviation to rotation about the x' axis

$$[\vec{\boldsymbol{\omega}}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \hat{\omega}'_1 \\ \hat{\omega}'_2 \\ \hat{\omega}'_2 \end{bmatrix}$$

where the $\hat{\omega}'_i \ll 1$ and are otherwise whatever you like. With a plot show the time histories of the $\hat{\omega}'_i$.

d) Do likewise with rotations about the y' and z' axis.

e) Conclude that axis about one or more of the axes is unstable and that the others are 'stable'. This is a numerical 'proof by example' of the linearized ODE demonstration given in lecture.

f) Using some non-trivial initial condition, show that the solution to these equations agrees with the solutions from the full equations with rotations. How to compare? Use the fixed frame $\vec{\omega}$ components and $\underline{\underline{\mathbf{R}}}$ as a function of time, from the previous problem, to find the body-fixed components of $\vec{\omega}$ as a function of time.

53. **Euler Equations and then some.** We have the great Euler equations for a rigid object:

$$\vec{M}_{/G} = \underline{\underline{\mathbf{I}}} \cdot \dot{\vec{\omega}} + \vec{\omega} \times \left(\underline{\underline{\mathbf{I}}} \cdot \vec{\omega} \right).$$

a) Write the Euler equations entirely in terms of components and or column vectors that are entirely in the $\mathcal{B}$ basis. For simplicity assume that $[\underline{\underline{\mathbf{I}}}]^{\mathcal{B}}$ is diagonal.

b) Set these up to solve in Matlab.

c) Using these equations for the evolution of $\vec{\omega}$, or the ones from the previous body-fixed representation, set up to solve for $\underline{\underline{\mathbf{R}}}(t)$ by setting up the evolution equations for $\underline{\underline{\mathbf{R}}}$, namely

$$\dot{\underline{\underline{\mathbf{R}}}} = \underline{\underline{\mathcal{S}}}(\vec{\omega}) \cdot \underline{\underline{\mathbf{R}}}$$

with matrix form in the $\mathcal{F}$ frame. Alternatively, you can assemble the rotation matrix as

$$R^{\mathcal{F}} = \left[\ [\hat{e}_1']_{\mathcal{F}} \quad [\hat{e}]_2']_{\mathcal{F}} \quad [\hat{e}_3']_{\mathcal{F}} \ \right] \qquad \text{and update with} \qquad \dot{\hat{\mathbf{e}}}_i' = \vec{\omega} \times \hat{\mathbf{e}}_i'.$$

d) Solve these in Matlab. Animate the motion using some pictorial representation you like (e.g. a box, ellipsoid or Jack) for these problems.

i) Pick some object and initial conditions that interest you.

ii) Consider an initial condition near to rotation nearly about the intermediate axis.

iii) Wobbling of a plate (See problem 56).

In each case you may want to make other plots, or show the trajectories of some special points, in order to clarify some statement you want to make (like about the relative rotation rates of the plate normal and of the whole plate).

e) Instead of representing the rotation with a rotation matrix use Euler angles. That is write the equations of motion using Euler angles and show at least one example where your solution agrees with the rotation matrix solution.

54. **Static and Dynamic Balance.** A rigid object is mounted rigidly to a fixed axis that has bearings *on* the x axis. That is,

$$\vec{\omega} = \omega \hat{\imath} = \text{constant.} \neq \vec{\mathbf{0}}$$

"Balance" in this context means that the net force $\vec{F}^{tot}$ and moment $\vec{M}_{\mathrm{C}}^{tot}$ are zero

$$\vec{F}^{tot} = \vec{\mathbf{0}} \qquad \text{and} \qquad \vec{M}_{\mathrm{C}}^{net} = \vec{\mathbf{0}}.$$

These are the total force on the object from the bearings and from any other forces or moments applied. That is, in a state of 'balance', the bearing reactions exactly cancel any other applied forces and moments, and nothing more. In these equations, the moment can be measured relative to any convenient fixed point C, typically one uses a point on the axis.

a) *Static* balance means $\vec{\boldsymbol{F}}^{net} = \vec{\mathbf{0}}$. Name precise conditions on the type of object, or location or orientation of the object that make it so static balance is achieved. Hint: (separate from looking at lecture notes) Use Linear Momentum Balance.

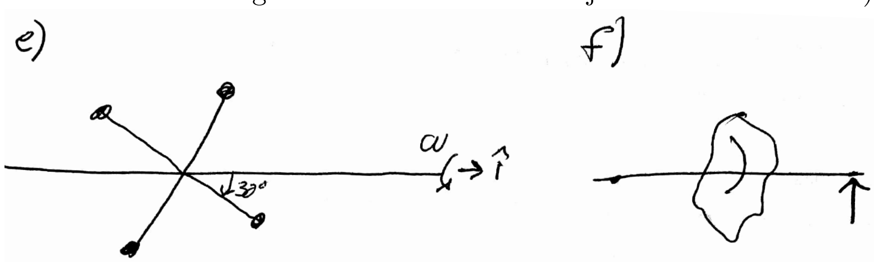
b) Describe a simple experiment to detect static balance. You can assume that you can disconnect the motor that causes $\vec{\boldsymbol{\omega}}$ and that the bearings are very low friction.

c) *Dynamic* balance means that both $\vec{\boldsymbol{F}}^{net} = \vec{\mathbf{0}}$ and $\vec{\boldsymbol{M}}^{net}_{\mathrm{C}} = \vec{\mathbf{0}}$. Describe precise conditions, in terms of the type of object, position of the object, or orientation of the object, so that dynamic balance is achieved.

d) Consider the most unsymetric object you can. For how many different positions and orientations can this object spin with dynamic balance? Describe them.

e) Consider this object. A rigid massless plus sign has four unit masses at the points ± 1 on the x and y axis. It is rotated 30°counter-clockwise about the z axis. This object is spun about the x axis with $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\imath}} =$ constant. $\neq \vec{\mathbf{0}}$. Show that this is dynamically balanced two ways: I. Using the moment of inertia, and II. adding up the moments and forces due to each of the 4 masses. Hint: for point C use the origin, the middle of the rotated plus sign.

f) Consider an unknown object spinning on a shaft at constant rate. The object is entirely between the bearings (totally between two parallel planes, both planes orthogonal to the axis and each plane passing through a bearing). It is claimed that there is only a reaction force at one end of the shaft, and none at the other. Is this possible? If not, prove it. If so, find an object where this is possible (pick dimensions for the bearing locations and the the object mass distribution).



55. **Spinning Satellite.**

Compare three satellites:

* All three Satelllites consists of uniform-density ($\rho = 1$) objects welded together: a unit cube and spheres each with radius 0.5.
* The spheres are welded to the middles of cube faces, just touching at one point.
* In the defining configuration, the cube corners are at $x, y, z = 0$ or 1.
* All satellites have a massless antennas, which go from (0,0,0) to (3,3,3) and an the other from (0,0,0.5) to (2,2,0.5).

(a) Satellite A has two spheres welded to the surfaces with outward normals in the $+x$ and $+y$ directions;
(b) Satellite B has a the third sphere on the $+z$ face; and
(c) Satellite C has spheres on all 6 faces.

Things to do:

i) Find the centers of mass of the sattelites.

ii) Find the moments of inertia matrix relative to the center of mass.

iii) Find the three principle moments of inertia (2 are the same for satellite A, why?)

iv) Animate these satellites tumbling in space (take care that the centers of mass are stationary).

I: rotating about each principle axis

II: The most arbitrary motion you can find. Look at the motion of the antenna tips. Can you describe this motion in a simple way?

III: If the answer to the problem above is in any way simple, can you find it by pencil and paper reasoning?

56. See **this video** from minute 10:05 to 11:03. Does Feynman have it right? That is your HW problem. At a minimum, use your 3D simulator to see if you get the motion he describes. To check this you have to compare the rate of wobbling of a normal to the plate to the rate of rotation of a drawing on the plate. If you want to do something analytical, assume a steady precessing solution of a flat plate. Keep clear in your head the relation between these three angular velocities: precession, spin and total.

57. **Spinning Satellite: Part II.** This problem picks up on the observations in problem 55 and asks for a bit more rigor. You might have already answered some of these questions when doing problem 55, and might not have got to them. Here is a chance to learn it all. Here are some claims about satellites, A,B and C from problem 55:

A) Satellite A.

i. Has three unequal principle moments of inertia, rated from biggest to smallest as 1, 2, and 3.
ii. The intermediate moment of inertia is parallel to the line connecting the centers of the two spheres.
iii. All of these are possible motions of the free satellite:
- exactly rotating about axis 1 or 2 or 3.
- wobbling about axis 1 or 3.
- wobbling about axis 2, but flipping occasionally. In this motion a directed line segment in the object, in the +2 direction, will wobble, the point approximately in the -2 direction, then wobble and then reverse.
- The motions above are all of the possible motions.

B) Satellite B.

i. Has two equal principle moments of inertia, so is dynamically axisymmetric.
ii. The symmetry axis is in the (1,1,1) direction.

iii. All of these are possible motions of the free satellite:
   - exactly rotating about the symmetry axis or about any axis perpendicular to the symmetry axis;
   - steady precession about the symmetry axis,
   - The motions above are all of the possible motions. For example, there is no wobbling about any other axis.

C) Satellite C.

i. Is dynamically a sphere; every direction is a principle direction.

ii. No matter what the initial condition, the motion is always with constant $\vec{\boldsymbol{\omega}}$. There are no precessing motions.

Rate all of the statements above as true or false. Justify your answer using computer calculations or simulations, or using reasoning based on the equations of mechanics. Be as convincing as you can be, within the time you want to allot to the problem. One useful thing is to track the trajectory of the tip of an appropriate line segment that is fixed in the satellite (say, the tip of one of the specified antennae, or some other antenna that you add). You don't need to hand in a simulation for each and every case. You can simply state what you observed in the simulation using clear-enough words that your answer seems convincing.

58. **Steady precession is general for an axisymmetric object in space?** This reinforces one point made in problem 57. If you covered it well enough already, you can skip this. For an axisymmetric object floating in space we found steady precessional motions. Are these the most general motions of an axisymmetric object in space? Yes or no. Make a convincing argument, numerical, analytical, or abstract, for your answer.

59. **Steady precession of a top.** Assume an axisymmetric top with mass m and principle inertias $I_1^G = I_2^G$ with I_3^G arbitrary but non-zero. The center of mass G is a distance d from the stationary pivot at O. This problem concerns steady precession solutions with $\hat{\boldsymbol{e}}_3'$ making a constant angle θ with the upwards $\hat{\boldsymbol{k}}$ direction. The top symmetry axis $\hat{\boldsymbol{e}}_3'$ precesses with constant rate ω_p around the $\hat{\boldsymbol{k}}$ axis. Relative to the precessing reference frame, the top spins with ω_s about the $\hat{\boldsymbol{e}}_3'$ axis. That is, at all times the angular velocity of the $\mathcal{T}$op is

$$\vec{\boldsymbol{\omega}}_{\mathcal{T}/\mathcal{F}} = \omega_p \hat{\boldsymbol{k}} + \omega_s \hat{\boldsymbol{e}}_3'$$

(a) Find steady precession motions of this top. That is, given g, d, m, I_1^G and I_3^G what restriction(s) do the laws of mechanics put on θ, ω_p and ω_s.

(b) Assume such precession is very close to upright ($\theta \ll 1$). What is the slowest spin, for given mass and inertia parameters, for which such steady precession solutions exist. This is a non-rigorous guess for the critical spin speed for upright stability.

(c) Are these the most general motions for an axisymmetric top with gravity? You can get a definitive answer to this question using your existent simulator, just adding in a torque from gravity.

60. **Steady precession of a Disk on plane with gravity.** We assume here steady motions with constant lean angle θ and constant pitch rate ω_s. In the steady motion

the center of mass of the disk traverses a circle with radius R. Given disk radius r, mass m, gravity g and inertia matrix in the disk frame, with $\hat{\boldsymbol{e}}_3'$ orthogonal to the disk plane,

$$\begin{bmatrix} I_1 = I & 0 & 0 \\ 0 & I_2 = I & 0 \\ 0 & 0 & I_3 = 2I \end{bmatrix}.$$

The laws of mechanics, with the contact conditions, give restrictions on the radius of the circular motions R, ω_s, the precession rate ω_p, and the lean angle ϕ. Find these for these cases:

(a) All solutions with no slip

(b) All solutions with no friction.

(c) The intersection of the two (all solutions with no slip and no friction), these are the most famous ones.

61. **Spinning top in 3D.** Find, solve and animate solutions for a spinning top in 3D. Use Euler angles. Check that, when used as initial conditions, the steady precession solutions (above) yield steady precession. Is the critical speed for the steady precessing solution indeed the minimum stable upright speed for the full 3D solutions? Investigate this either numerically, analytically, or both.